Poglavje 1

OSNOVNE ENAČBE TEORIJE ELASTIČNOSTI

I. Prostorsko napetostno-deformacijsko stanje
Kartezijev koordinatni sistem

1. Vektor pomikov:

$$u_i = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

2. Deformacijski tenzor:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

3. Napetostni tenzor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

4. Sovisnosti med deformacijami in pomiki:

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u_x}{\partial x} \\
\epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\epsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\
\epsilon_{yy} &= \frac{\partial u_y}{\partial y} \\
\epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\gamma_{xy} &= 2\epsilon_{xy} \\
\gamma_{yz} &= 2\epsilon_{yz} \\
\gamma_{xz} &= 2\epsilon_{xz}
\end{aligned}
$$

5a. Sovisnosti med deformacijami in napetosti:

$$\epsilon_{ij} = \frac{1}{2G}\,\sigma_{ij} - \frac{\nu}{E}\,\sigma_{kk}\delta_{ij}$$

$$
\begin{aligned}
\epsilon_{xx} &= \frac{1}{2G}\,\sigma_{xx} - \frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \\
\epsilon_{xy} &= \frac{1}{2G}\,\sigma_{xy} \\
\epsilon_{xz} &= \frac{1}{2G}\,\sigma_{xz} \\
\epsilon_{yy} &= \frac{1}{2G}\,\sigma_{yy} - \frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \\
\epsilon_{yz} &= \frac{1}{2G}\,\sigma_{yz} \\
\epsilon_{zz} &= \frac{1}{2G}\,\sigma_{zz} - \frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)
\end{aligned}
$$

$$G = \frac{E}{2\left(1 + \nu\right)}$$

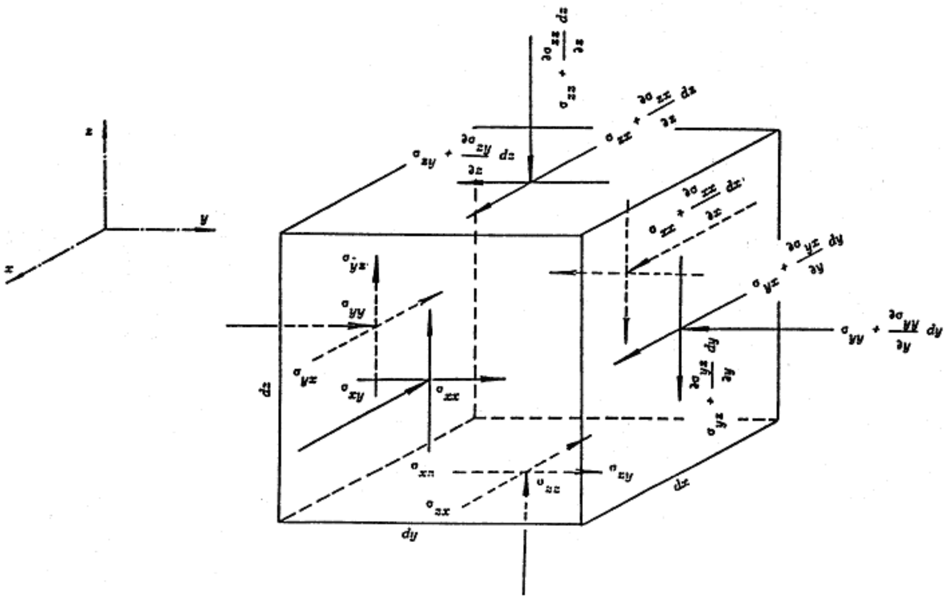
5b. Sovisnosti med napetostmi in deformacijami:

$$\sigma_{ij} = 2G\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\delta_{ij}$$

$$\begin{aligned}
\sigma_{xx} &= 2G\,\epsilon_{xx} + \lambda\,(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \\
\sigma_{xy} &= 2G\,\epsilon_{xy} = G\,\gamma_{xy} \\
\sigma_{xz} &= 2G\,\epsilon_{xz} = G\,\gamma_{xz} \\
\sigma_{yy} &= 2G\,\epsilon_{yy} + \lambda\,(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \\
\sigma_{yz} &= 2G\,\epsilon_{yz} = G\,\gamma_{yz} \\
\sigma_{zz} &= 2G\epsilon_{zz} + \lambda\,(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})
\end{aligned}$$

$$\lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)}$$

6. Ravnovesne enačbe:



Slika 1.1: Prerez prizme

$$\sigma_{ij,j} + f_i = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

II. Osnosimetrično napetostno-deformacijsko stanje
Cilindrični koordinatni sistem

1. Vektor pomikov:

$$u_\alpha = \begin{Bmatrix} \rho \\ w \end{Bmatrix}$$

2. Deformacijski tenzor:

$$\epsilon_{\alpha\beta} = \begin{bmatrix} \epsilon_r & 0 & \epsilon_{rz} \\ 0 & \epsilon_t & 0 \\ \epsilon_{zr} & 0 & \epsilon_z \end{bmatrix}$$

3. Napetostni tenzor:

$$\sigma_{\alpha\beta} = \begin{bmatrix} \sigma_r & 0 & \sigma_{rz} \\ 0 & \sigma_t & 0 \\ \sigma_{zr} & 0 & \sigma_z \end{bmatrix}$$

4. Sovisnosti med deformacijami in pomiki:

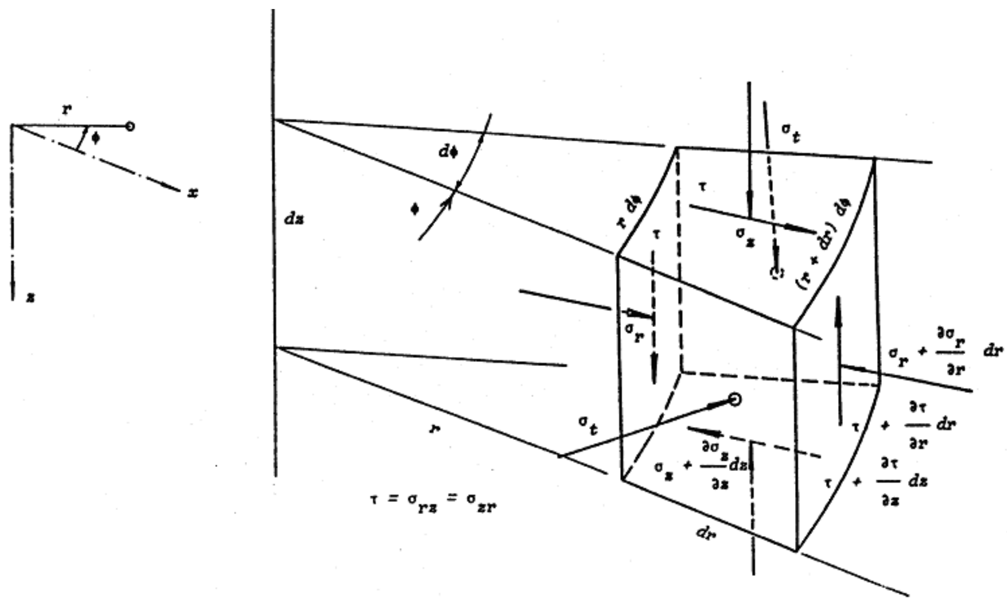
$$\begin{aligned} \epsilon_r &= \frac{\partial \rho}{\partial r} \\ \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial \rho}{\partial z} + \frac{\partial w}{\partial z}\right) \\ \epsilon_t &= \frac{\rho}{r} \\ \epsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{rz} &= 2\,\epsilon_{rz} \end{aligned}$$

$$\epsilon_v = \epsilon_r + \epsilon_t + \epsilon_z = \frac{\partial \rho}{\partial r} + \frac{\rho}{r} + \frac{\partial w}{\partial z}$$

5. Sovisnosti med napetostmi in deformacijami:

$$
\begin{aligned}
\sigma_r &= \frac{E}{(1+\nu)}\left[\epsilon_r + \frac{\nu}{(1-2\nu)}\,\epsilon_v\right] \\
\sigma_{rz} &= \tau = \frac{E}{(1+\nu)}\,\epsilon_{rz} \\
\sigma_t &= \frac{E}{(1+\nu)}\left[\epsilon_t + \frac{\nu}{(1-2\nu)}\,\epsilon_v\right] \\
\sigma_z &= \frac{E}{(1+\nu)}\left[\epsilon_z + \frac{\nu}{(1-2\nu)}\,\epsilon_v\right]
\end{aligned}
$$

6. Ravnovesni enačbi:



Slika 1.2: Prerez prizme

$$
\sigma_r + \frac{\partial \sigma_r}{\partial r}\, r - \sigma_t + \frac{\partial \tau}{\partial z}\, r + f_r = 0
$$

$$
\frac{\partial \sigma_z}{\partial z}\, r + \tau + \frac{\partial \tau}{\partial r}\, r + f_z = 0
$$

1. Lammejevi ravnovesni enačbi:

   Izrazimo napetosti s pomiki:

$$
\begin{aligned}
\sigma_r &= \frac{E}{1+\nu}\left[\frac{\partial\rho}{\partial r}+\frac{\nu}{1-2\nu}\left(\frac{\partial\rho}{\partial r}+\frac{\rho}{r}+\frac{\partial w}{\partial z}\right)\right] \\
\sigma_{rz} &= \frac{E}{1+\nu}\,\frac{1}{2}\left(\frac{\partial\rho}{\partial z}+\frac{\partial w}{\partial z}\right) \\
\sigma_t &= \frac{E}{1+\nu}\left[\frac{\rho}{r}+\frac{\nu}{1-2\nu}\left(\frac{\partial\rho}{\partial r}+\frac{\rho}{r}+\frac{\partial w}{\partial z}\right)\right] \\
\sigma_z &= \frac{E}{1+\nu}\left[\frac{\partial w}{\partial z}+\frac{\nu}{1-2\nu}\left(\frac{\partial\rho}{\partial r}+\frac{\rho}{r}+\frac{\partial w}{\partial z}\right)\right]
\end{aligned}
$$

   V ravnovesnih enačbah upoštevamo izraze za napetosti, izražene s pomiki. Po ureditvi dobimo:

$$\frac{\partial^2\rho}{\partial z^2}+2\,\frac{\partial^2\rho}{\partial r^2}+\frac{2}{r}\frac{\partial\rho}{\partial r}+\frac{\partial^2 w}{\partial r\partial z}-\frac{2\rho}{r^2}+\frac{2\nu}{1-2\nu}\frac{\partial\epsilon_v}{\partial r}=0$$

$$2\,\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2\rho}{\partial r\partial z}+\frac{1}{r}\frac{\partial\rho}{\partial z}+\frac{2\nu}{1-2\nu}\frac{\partial\epsilon_v}{\partial z}=0$$

Upoštevamo Laplaceov operator ∇, ki ima v cilindičnih koordinatah naslednji pomen:

$$\nabla=\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$$

Po ureditvi dobimo Lammejevi ravnovesni enačbi:

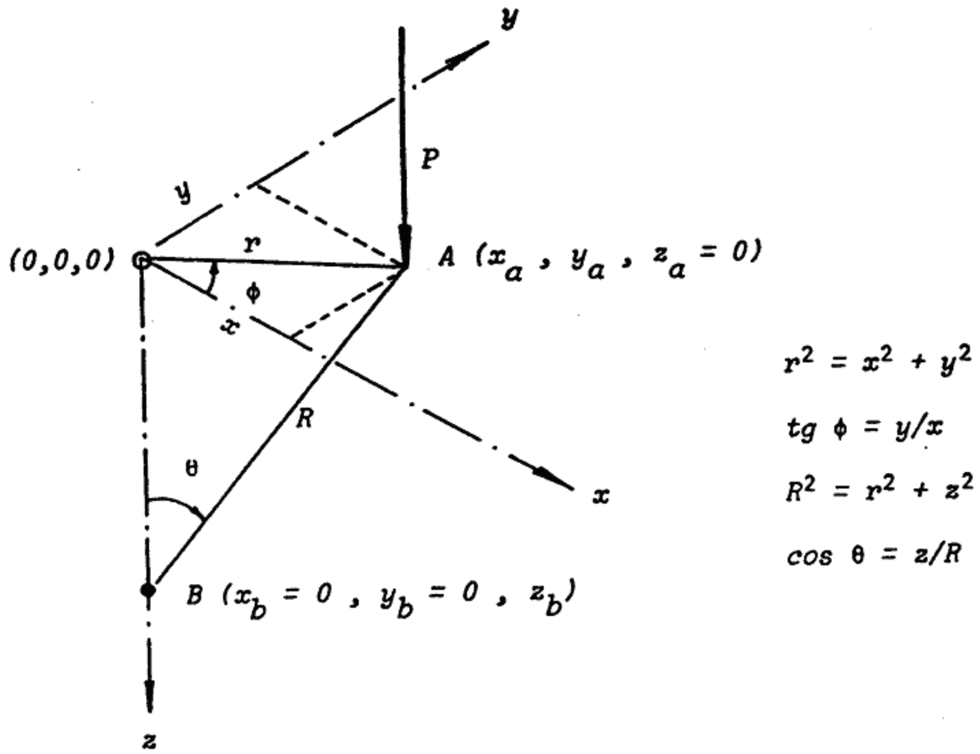
$$\nabla\rho-\frac{\rho}{r^2}+\frac{1}{1-2\nu}\frac{\partial\epsilon_v}{\partial r}=0$$

$$\nabla w+\frac{1}{1-2\nu}\frac{\partial\epsilon_v}{\partial z}=0$$

III. Boussinesqova rešitev Lammejevih enačb za polprostor obremenjen z vertikalno koncentrirano silo P.

Ravnovesni enačbi:

$$\nabla\rho-\frac{\rho}{r^2}+\frac{1}{1-2\nu}\frac{\partial\epsilon_v}{\partial r}=0$$



Slika 1.3: Prerez prizme

$$\nabla w + \frac{1}{1-2\nu}\frac{\partial \epsilon_v}{\partial z} = 0$$

Robni pogoji:

1. $r \to \infty$: vse napetosti so nič
2. $r \to \infty$: vse deformacije so nič.
3. $z = 0$: $\tau = 0$
4. $z = 0$: $\sigma_z = 0$, razen pod koncentrirano silo

Ravnovesni pogoj:

Rezultanta napetosti, ki učinkuje na površino polkrogle, obdajajoče prijemališče vertikalne koncentrirane sile, mora biti nasprotno usmerjena po smernici vertikalne sile in po velikosti enaka vertikalni koncentrirani sili.

Pomiki:

$$\begin{aligned}
\rho &= \frac{P}{4\pi RG}\left[\sin\vartheta\,\cos\vartheta - (1-2\nu)\,\frac{\sin\vartheta}{1+\cos\vartheta}\right] \\
w &= \frac{P}{4\pi RG}\left[2(1-\nu) + \cos^2\vartheta\right]
\end{aligned}$$

Napetosti:

$$\begin{aligned}
\sigma_r &= \frac{P}{2\pi R^2}\left(3\sin^2\vartheta\,\cos\vartheta - \frac{1-2\nu}{1+\cos\vartheta}\right) \\
\tau &= \frac{3P}{2\pi R^2}\,\sin\vartheta\, cos^2\vartheta \\
\sigma_t &= \frac{P\,(1-2\nu)}{2\pi R^2}\left(\frac{1}{1+\cos\vartheta} - \cos\vartheta\right) \\
\sigma_z &= \frac{3P}{2\pi R^2}\,\cos^3\vartheta
\end{aligned}$$

IV. Transformacija vektorjev in tenzorjev iz enega koordinatnega sistema v drug koordinatni sistem.

$$u_i = u_\alpha\, e_{\alpha i}$$

$$\sigma_{ij} = \sigma_{\alpha\beta}\, e_{\alpha i}\, e_{\beta j}$$

Matrika smernih kosinusov za transformacijo iz cilindričnega v kartezijev koordinatni sistem:

$$e_{\alpha i} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pomiki:

$$\begin{aligned}
u_x &= \rho\cos\varphi \\
u_y &= \rho\sin\varphi \\
u_z &= w
\end{aligned}$$

Napetosti:

$$
\begin{aligned}
\sigma_{xx} &= \sigma_r \cos^2\varphi + \sigma_t \sin^2\varphi \\
\sigma_{xy} &= (\sigma_r - \sigma_t)\sin\varphi \cos\varphi \\
\sigma_{xz} &= \tau \cos\varphi \\
\sigma_{yy} &= \sigma_r \sin^2\varphi + \sigma_t \cos^2\varphi \\
\sigma_{yz} &= \tau \sin\varphi
\end{aligned}
$$